BRANCHING OF ROTATIONALLY SYMMETRIC SOLUTIONS DESCRIBING FLOWS OF A VISCOUS LIQUID WITH A FREE SURFACE

V. V. Pukhnachev

UDC 532.516

Equilibrium shapes of a liquid, situated on the outer or inner surface of a rigid cylinder and rotating together with it as a solid body, are studied. We determine the principal part of the solution of the equilibrium equation for small deviations of the determining parameter from the critical value. The bifurcation of rotationally symmetric motions with a free boundary in a body force field is also investigated.

1. Suppose that a viscous, incompressible, weightless, capillary liquid fills the space between two neighboring cylindrical surfaces of radii r_1 and $r_2 > r_1$. The inner surface is rigid and rotates about its axis with an angular velocity Ω. We introduce dimensionless parameters, choosing as scales of length, velocity, and pressure the quantities r_2, Ωr_2, $\rho\Omega^2 r_2^2$ (ρ being the density of the liquid). We shall discuss the motion in a cylindrical coordinate system r, θ, z, where $r=0$ is the common axis of the cylinders.

We assume that a rotational field of body forces $[0, F(r), 0]$ acts on the liquid. Then the Navier–Stokes equations have a steady solution in which the velocity field is $[0, V(r), 0]$ and the pressure is

$$P(r) = -\int_r^1 \frac{V^2(s)}{s}\, ds + P_*$$

where P_* is a constant. The function $V(r)$ is uniquely determined as the solution of the boundary problem

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{1}{r^2}V = -F(r)$$

$$V = a \quad \text{for} \quad r = a, \qquad \frac{dV}{dr} - \frac{1}{r}V = 0 \quad \text{for} \quad r = 1$$

$$(a = r_2 / r_1 < 1).$$

At the same time the adhesion condition holds on the surface $r=a$, while the conditions of impenetrability and lack of tangential stress hold on the surface $r=1$. Choosing $P_* = \beta^{-1}$, where $\beta = \rho\Omega^2 r_2^3/\sigma$, σ being the surface tension coefficient, we can verify that at $r=1$ the normal stress is equal to the capillary pressure. Thus the surface $r=1$ is a free surface.

Of physical interest is the case $F=0$, so that

$$V = r, \quad P = (r^2 - 1)/2 + \beta^{-1}.$$

In this case the liquid and the cylinder rotate as one solid body.

We shall call the motion of the liquid described above fundamental. Later we shall look for rotationally symmetric motions that branch off from the fundamental motion. We seek velocity and pressure fields of the form

$$\mathbf{v}^1 = (u, V + v, w), \quad p^1 = P + R^{-1}p \; (R = \Omega r_2^2 / \nu),$$

where R is the Reynolds number. The functions u, v, w, p depend only on r, z. Insertion of $\mathbf{v}^1$, p^1 into the Navier–Stokes equations results in the following system, involving $\mathbf{v} = (u, v, w)$ and p:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 127–134, March–April, 1973. Original article submitted November 11, 1972.

$$\begin{aligned}\Delta u - r^{-2}u - p_r + 2R\omega v &= R\,(uu_r + wu_z - r^{-1}v^2)\\ \Delta v - r^{-2}v - Rgu &= R\,(uv_r + wv_z + r^{-1}uv)\\ \Delta w - p_z &= R\,(uw_r + ww_z)\\ r^{-1}(ru)_r + w_z &= 0\end{aligned} \tag{1.1}$$

Here the subscripts r, z denote corresponding partial derivatives and

$$\omega(r) = \frac{V}{r}, \qquad g(r) = \frac{dV}{dr} + \frac{V}{r}$$

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Let $r = \eta(z)$ represent the equation of the free surface. It will be assumed that the motion is periodic with period l in the z-direction:

$$\mathbf{v}(r, z + l) = \mathbf{v}(r, z), \quad p(r, z + l) = p(r, z), \quad \eta(z + l) = \eta(z) \tag{1.2}$$

and that u, v, p, η are even functions of z, while w is odd:

$$\begin{aligned} u(r, z) &= u(r, -z), \quad v(r, z) = v(r, -z), \quad \eta(z) = \eta(-z)\\ p(r, z) &= p(r, -z), \quad w(r, z) = -w(r, -z)\end{aligned} \tag{1.3}$$

On the rigid surface the adhesion condition holds:

$$u = v = w = 0 \ \text{ for } \ r = a \tag{1.4}$$

On the free boundary the normal velocity component and the tangential stress vanish, while the normal stress is equal to the capillary pressure. In terms of $\mathbf{v}$, p, η this can be written

$$u - \dot\eta w = 0 \ \text{ for } \ r = \eta \tag{1.5}$$

$$(1 - \dot\eta^2)(u_z + w_r) + 2\dot\eta(u_r - w_z) = 0 \ \text{ for } \ r = \eta \tag{1.6}$$

$$(V + v)_r - r^{-1}(V + r) - \dot\eta v_z = 0 \ \text{ for } \ r = \eta \tag{1.7}$$

$$\frac{1}{\beta}\left[\frac{\ddot\eta}{(1 + \dot\eta^2)^{3/2}} - \frac{1}{\eta(1 + \dot\eta^2)^{1/2}}\right] = -\left(P + \frac{p}{R}\right) + \frac{2}{R(1 + \dot\eta^2)}\left[u_r - \dot\eta^2(u_z + w_r) + \dot\eta^2 w_z\right] \ \text{ for } \ r = \eta$$

$$(\dot\eta = d\eta/dz, \quad \ddot\eta = d^2\eta/dz^2) \tag{1.8}$$

To these conditions we add a condition on the constancy of the volume of the liquid contained between the planes $z = 0$ and $z = l$ in the fundamental and the perturbed motions

$$\int_0^l (\eta^2 - 1)\,dz = 0 \tag{1.9}$$

Equations (1.1) with the conditions (1.2)-(1.9) have the trivial solution $\mathbf{v} = 0$, $p = 0$, $\eta = 1$ (we recall that $V'(1) - V(1) = 0$, $P(1) = \beta^{-1}$). We shall find solutions of the problem (1.1)-(1.9) that branch off from the trivial solution for some values of the parameters β, R.

2. In this and the following sections it is assumed that $F = 0$, $V = r$, while $w = 1$, $g = 2$. We shall show that then $v = 0$, $p = \text{const}$ in the solution of the problem (1.1)-(1.9).

We multiply the first of Eqs. (1.1) by u, the second by v, and the third by w. We then integrate these equalities over the range $0 < z < l$, $a < r < \eta$ and add the results. Both parts of the relationship obtained are integrated by parts with the use of the equation of continuity and conditions (1.2)-(1.7). We find

$$\int_0^l dz \int_a^{\eta(z)} \left[u_r^2 + \frac{u^2}{r^2} + w_z^2 + \frac{1}{2}\left(v_r - \frac{v}{r}\right)^2 + \frac{1}{2}v_z^2 + \frac{1}{2}(u_z + w_r)^2\right] r\,dr$$

From this and from (1.4) it follows that $u = v = w = 0$. Equation (1.1) gives $p = \text{const}$; we denote this constant by C.

Thus, if the fundamental solution is the rotation of the liquid as a solid body, then in the perturbed motion the velocity field remains as it was, while only the free surface is perturbed. This fact was noted by Chia-Shun Yih [1], who has studied the stability of a liquid film on the surface of a rotating cylinder in a linear approximation. The branching problem reduces to the search for a function η and a constant C for which the following equation is satisfied

$$\frac{\ddot\eta}{(1 + \dot\eta^2)^{3/2}} - \frac{1}{\eta(1 + \dot\eta^2)^{1/2}} + 1 + \frac{\beta}{2}(\eta^2 - 1) + \frac{\beta C}{R} = 0, \tag{2.1}$$

along with condition (1.9) and the condition that η be a periodic even function.

The problem we have formulated is a special case of a problem considered by L. A. Slobozhanin [2]. In this paper the branching problem we investigated by the Lyapunov–Schmidt method in the case where a liquid is contained between parallel plates $z=0$ and $z=l/2$ and rotates together with them as a rigid body. The equilibrium shape was not assumed to be axisymmetric ahead of time. The plane analogue of this problem was considered earlier by Yu. K. Bratukhin and L. N. Maurin [3].

By use of a method suggested in [2], the problem (2.1), (1.9) is reduced to an integrodifferential equation for η. To achieve this one must eliminate the constant C, integrate (2.1) from 0 to l, and make use of (1.9). As the result of this we obtain

$$\frac{\ddot{\eta}}{(1+\dot{\eta}^2)^{3/2}}-\frac{1}{\eta(1+\dot{\eta}^2)^{1/2}}+\frac{\beta}{2}(\eta^2-1)+\frac{1}{l}\int_0^l\frac{dz}{\eta(1+\dot{\eta}^2)^{1/2}}=0 \tag{2.2}$$

Equation (2.2), linearized near $\eta=1$, has the form

$$\ddot{x}+x-\frac{1}{l}\int_0^l x\,dz+\beta x=0 \tag{2.3}$$

Uniting the conditions

$$x(z+l)=x(z),\ x(z)=x(-z)$$

with (2.3) we obtain a linear problem for the eigenvalues. Its eigenvalues are

$$\beta_0=0,\ \beta_k=\alpha^2k^2-1\quad\text{for}\quad k=1,\ 2,\ \ldots\ (\alpha=2\pi/l),$$

and the eigenfunctions are as follows:

$$x_0=1,\ x_k=\cos\alpha kz\ \text{for}\quad k\geqslant 1$$

Each proper number is simple and, according to a theorem by M. A. Krasnocel'skii [4], there is a bifurcation point of Eq. (2.2). Keeping in mind that the parameter $\beta=\rho\Omega^2r_2^3/\sigma$ must be positive to have physical meaning, we write the branching condition in the form

$$\rho\Omega^2r_2^3/\rho=(2\pi k/l)^2-1,\quad 2\pi k/l>1\quad (k=1,2,\ldots). \tag{2.4}$$

We note that the branching condition does not depend on the Reynolds number. This fact was mentioned in [1].

To find the principal parts of solutions of Eq. (2.2) that branch from $\eta=1$ we use the Lyapunov–Schmidt method. Following [2], we can show that near $\eta=1$ real solutions of Eq. (2.2) exist only for $\beta<\beta_k$. (A similar deduction was made in [3].) Moreover, we shall restrict our search to solutions branching for the smallest bifurcation value; $\beta=\beta_1=\alpha^2-1$; $\alpha>1$ so that $\beta_1>0$. Putting $\mu^2=\alpha^2-1-\beta$, we seek a solution of the form

$$\eta=1+\mu\eta_1+\mu^2\eta_2+\mu^3\eta_3+\ldots$$

(The convergence of this series follows from results of [2].) The functions η_1, η_2, η_3 satisfy the equations

$$\ddot{\eta}_1+\alpha^2\eta_1-\frac{1}{l}\int_0^l\eta_1\,dz=0 \tag{2.5}$$

$$\ddot{\eta}_2+\alpha^2\eta_2-\frac{1}{l}\int_0^l\eta_2\,dz=-\frac{\alpha^2-3}{2}\eta_1^2-\frac{1}{2}\dot{\eta}_1^2+\frac{1}{l}\int_0^l\left(-\eta_1^2+\frac{1}{2}\dot{\eta}_1^2\right)dz \tag{2.6}$$

$$\ddot{\eta}_3+\alpha^2\eta_3-\frac{1}{l}\int_0^l\eta_3\,dz=\eta_1-(\alpha^2-3)\eta_1\eta_2-\dot{\eta}_1\dot{\eta}_2-\eta_1^3$$

$$-\frac{3\alpha^2-1}{2}\eta_1\dot{\eta}_1^2+\frac{1}{l}\int_0^l\left(-2\eta_1\eta_2+\dot{\eta}_1\dot{\eta}_2+\eta_1^3-\frac{1}{2}\eta_1\dot{\eta}_1^2\right)dz \tag{2.7}$$

Equations (2.5) and (2.6) are solved sequentially and give

$$\eta_1=q\cos\alpha z,\quad \eta_2=-{}^1/_4q^2+q^1\cos\alpha z-{}^1/_4q^2\alpha^{-2}\cos 2\alpha z$$

where q, q^1 are undetermined coefficients. The constant q is found from the condition that Eq. (2.7) be solvable in the class of l-periodic functions. Its value is

$$q = 2^{3/2} 3^{-1/2} \alpha (\alpha^6 - \alpha^4 + 3\chi^2 + 1)^{-1/2} \quad (2.8)$$

(we discard the second value of q because the corresponding solution of Eq. (2.2) is obtained from the solution at hand by replacing z with $z + l/2$.) The constant q^1 is found from the solvability condition on the equation for η_4, which will not be given. It is found that $q^1 = 0$. Thus the principal part of the solution of (2.2) for small $\mu = (\alpha^2 - 1 - \beta)^{1/2}$ is represented in the form

$$\eta = 1 + \mu q \cos \alpha z - \mu^2 q^2 ({}^1/_4 + {}^1/_4 \alpha^{-2} \cos 2\alpha z) + O(\mu^3) \quad (2.9)$$

where $q(\alpha)$ is given by Eq. (2.8).

3. The branching condition (2.4), as well as the shape of the perturbed surface, does not depend on the parameter a, the dimensionless radius of the rigid cylinder. (It is assumed that μ is so small that $\min \eta > a$.) The solution of the problem on the axisymmetric equilibrium shapes of a rotating liquid that branch from a circular cylinder does not change when the inner body has an arbitrary shape, so long as it is contained in the cylinder $r \leq a < 1$. In particular, the rigid body can be absent altogether. Moreover, the above considerations remain in force if the equality $V = r$ for the fundamental motion holds only for r in some neighborhood of unity. If the wetting angle is equal to $\pi/2$, the solution of the problem can be combined with the conditions for adherence on planes $z = 0$ and $z = kl/2$ that rotate together with the cylinder; we have a problem similar to that considered in [2].

It has already been pointed out above that the branching condition (2.4) does not depend on the Reynolds number. Thus nonuniqueness in the equilibrium shape of a rotating liquid appears for arbitrarily small Reynolds numbers. The origin of this effect is the presence of the free surface. It is well known that at small Reynolds numbers the steady motion of a viscous liquid, bounded by rigid walls, is unique. The nonuniqueness of a motion with a free boundary, described above, is essentially of a geometrical nature; it is connected with the onset of "deflection" of the minimal surface in the centrifugal force field, given by Eq. (2.1). This deformation of the surface is not isometric, but it conserves volume.

In the perturbed and the fundamental motions the pressure differs by the constant $C = C(\mu)$; $C \to 0$ for $\mu \to 0$. By subjecting the solution to the additional condition $C(\mu) \equiv 0$ we obtain a relationship between $\alpha = 2\pi/l$ and μ. The relationship $C = 0$, (2.1), and (1.9) lead to the equality

$$\frac{1}{l} \int_0^l \frac{dz}{\eta (1 + \dot{\eta}^2)^{1/2}} = 1$$

Inserting expression (2.9) into this equality, we find that $\alpha = \sqrt{3} + O(\mu)$ for $\mu \to 0$.

It is stated in [2] that for small μ branching equilibrium shapes are unstable. Results of [3] indicate the existence of equilibrium shapes that are remote from the trivial one (for the plane problem).

If, with increasing μ, the condition that the free boundary should not intersect the surface of the cylinder is violated, $\min \eta \geq a$, then the solution of Eq. (2.2) loses physical meaning. In this case, however, one can look for nontrivial equilibrium shapes of the rotating liquid with unconnected free boundaries. Finding them involves the determination of a piecewise-smooth function $\eta(z)$ which satisfies Eq. (2.2) only in the interval $|z| < b$ $(0 < b < l/2)$, while for $b < z < l/2$, $-l/2 < z < -b$ one has $\eta = a$. First of all the function η is assumed to be l-periodic, even, and such that condition (1.9) is satisfied. At points where the free boundary touches the rigid surface the conditions

$$\eta(\pm b) = a, \quad \dot{}\,(\pm b) = \mp \operatorname{tg} \gamma$$

are satisfied, where $\gamma \in (0, \pi)$ is a prescribed wetting angle. The parameter b is determined during the course of solution. A solution of the problem just formulated exists if the quantities $(1-a)/l$ and γ are sufficiently small.

We shall now consider the problem on the equilibrium shapes of a liquid situated on the inner surface of a hollow cylinder and rotating with it as a solid body. This problem reduces to Eq. (2.2), in which the parameter β must be replaced by $-\beta$. The bifurcation values of β are given by the formula $\beta_k = 1 - \alpha^2 k^2$, $k \geq 1$ being an integer. As $\beta > 0$, there are a finite number of them for any fixed value of $\alpha < 1$. A pair of equilibrium shapes branches from each value β_k that is less than β. This conclusion is somewhat unexpected, as it runs contrary to a priori notions concerning the stabilizing role of centrifugal force in the motion under consideration.

4. We shall now discuss the branching problem (1.1)-(1.9) in the case $V \not\equiv r$.

We denote by $C_l^{m+\lambda}$ the subspace of l-periodic even functions of the Hölder space $C^{m+\lambda}(-\infty, \infty)$, $m \geq 0$ being an integer, and $0 < \lambda < 1$. Let $\eta(z) \in C_l^{3+\lambda}$ and $\min \eta > a$. For fixed η and $V(r) \in C^3 \times [a, \max \eta]$ we consider the following auxiliary problem: to find in the interval $\pi = \{r, z: a \leqslant r \leqslant \eta(z), -\infty < z < \infty\}$ a solution $\mathbf{v}(r, z)$, $p(r, z)$ of the system (1.1) that satisfies conditions (1.2)-(1.7). In what follows C_0, C_1, C_2 denote positive constants and $|\mathbf{v}|_{2+\lambda}$, $|\nabla p|_\lambda$, $|\eta - 1|_{3+\lambda}$ denote Hölder norms of the corresponding functions, calculated over their domains of definition. The following proposition, stated without proof, is valid.

Proposition 4.1. Let $|\eta - 1|_{3+\lambda} \leqslant \varepsilon$, where $\varepsilon > 0$ is sufficiently small, and for $a < r < 1+\varepsilon$ let one of the following conditions hold:

$$\omega(r) = \frac{V}{r} > 0, \quad g(r) = \frac{dV}{dr} + \frac{V}{r} > 0 \tag{4.1}$$

$$R\left|\frac{dV}{dr} - \frac{V}{r}\right| < C_0 \tag{4.2}$$

where C_0 depends only on l, a, ε. Then the solution of the problem (1.1)-(1.7) exists uniquely (in the small) and satisfies the inequality

$$|\mathbf{v}|_{2+\lambda} + |\nabla p|_\lambda \leqslant C_1(|K|\,|\eta - 1|_{3+\lambda} + |\eta - 1|_{3+\lambda}^2) \qquad (K = V''(1)) \tag{4.3}$$

The assertion concerning the uniqueness of the solution of problem (1.1)-(1.7) must be understood in the following sense: the velocity vector $\mathbf{v}$ is determined unambiguously and the pressure p is determined to within an additive constant. We put $p = p_0 + C$, where the function p_0 is determined unambiguously for a given $\mathbf{v}$ by Eq. (1.1) and the condition

$$\int_0^l dz \int_a^{\eta(z)} p_0 r\, dr = 0$$

For simplicity it is further assumed that $V''(1) = K = 0$. In this case the branching condition for the problem (1.1)-(1.9) can be written explicitly.

We insert the solution $\mathbf{v}$, $p = p_0 + C$ of the auxiliary problem into the remaining condition on the free surface (1.8). In the equality (1.8) we replace $P(\eta)$ with $P = \beta^{-1} + s(\eta^2 - 1)/2 + \Phi$, where $s = V^2(1)$ and $\Phi(\eta) = O(\eta - 1)^3$ for $\eta \to 1$; we assume that $V(1) \neq 0$. We then transform the resulting relationship by eliminating the constant C by condition (1.9), obtaining

$$N(\eta) \equiv \frac{\ddot{\eta}}{(1 + \dot{\eta}^2)^{3/2}} - \frac{1}{\eta(1 + \dot{\eta}^2)^{1/2}} + 1 + \frac{\beta s}{2}(\eta^2 - 1) + \frac{1}{l}\int_0^l \frac{dz}{\eta(1 + \dot{\eta}^2)^{1/2}} + Q(\eta) - \frac{1}{l}\int_0^l Q[\eta(z)]\,dz = 0$$

$$\left(Q(\eta) = \beta\Phi(\eta) + \frac{\beta}{R}\bar{p}_0(\eta) - \frac{2\beta}{R(1 + \dot{\eta}^2)}\{\bar{u}_r(\eta) - \dot{\eta}[\bar{u}_z(\eta) + \bar{w}_r(\eta)] + \dot{\eta}^2\bar{w}_z(\eta)\}\right) \tag{4.4}$$

The symbol $\bar{u}_r$ denotes an operator acting according to the rule: $\bar{u}_r[\eta(z)] = u_r[\eta(z), z]$, where $u(r, z)$ is determined from the solution of the problem (1.1)-(1.7). The operators $\bar{u}_z, \ldots, \bar{p}_0$ are determined in a similar way.

We note that under the conditions of Proposition 4.1 we have from the inequality (4.3) and the definition of $K=0$ an estimate of Φ of the form $Q(\eta)$

$$|Q(\eta)|_{1+\lambda} \leqslant C_2|\eta - 1|_{3+\lambda}^2 \tag{4.5}$$

Relationship (4.4) can be treated as an operator equation for the determination of the function $\eta(z)$. If η is a solution of (4.4), the constant-volume condition (1.9) is satisfied automatically. As a consequence of the estimate (4.3) and conditions (1.2), (1.3), which the functions u, w, p satisfy, the operator N acts out of $C_l^{3+\lambda}$ into $C_l^{1+\lambda}$. To demonstrate this we perform a Fréchet differentiation on the operator N in the sphere $|\eta - 1|_{3+\lambda} < \varepsilon$. On the basis of (4.4), (4.5) its Fréchet derivative at the point $\eta = 1$ has the form

$$\ddot{x} + x - \frac{1}{l}\int_0^l x\,dz + \beta s x \equiv L(x) + \beta s x \tag{4.6}$$

The operator L has the simple proper numbers $\mu_0 = 0$, $\mu_k = 1 - \alpha^2 k^2$ for $k = 1, 2, \ldots$. As $\mu_k > 1$, there exists an operator that is the inverse of L−I, namely $(L - I)^{-1}$: $C_l^{1+\lambda} \to C_l^{3+\lambda}$ (I is the identity operator). From (4.4)-(4.6) and the definition of a Fréchet derivative it follows that

$$N(x + 1) = L(x) + \beta s x + T(x) \qquad (|T(x)|_{1+\lambda} = O(|x|_{3+\lambda}^2) \text{ for } x \to 0)$$

With the notation $x = \eta - 1$, we write Eq. (4.4) in the form

$$(L - I)(x) + T(x) = -(1 + \beta s)\, x$$

and we apply the operator $(L-I)^{-1}$ to both sides of the equality, obtaining

$$x + (L - I)^{-1} T(x) = -(1 + \beta s)(L - I)^{-1}(x) \tag{4.7}$$

Here the operator $(L-I)^{-1}$ is completely continuous in $C_l^{3+\lambda}$. The operator $(L-I)^{-1}T$ is continuous and admits of the estimate

$$|(L - I)^{-1} T(x)|_{3+\lambda} = O(|x|^2_{3+\lambda}) \text{ for } x \to 0.$$

It follows from this that there exists a resolvent of the nonlinear operator $(L-I)^{-1}T$. In other words, the equation $x + (L-I)^{-1}T(x) = f, f \in C_l^{3+\lambda}$ has a unique solution for any sufficiently small $|f|_{3+\lambda}$ in some sphere $|x|_{3+\lambda} < \delta$, $0 < \delta < \varepsilon$. We represent this solution in the form $x = \mathcal{R}f$, where $\mathcal{R}$ is a continuous operator in $C_l^{3+\lambda}$. Moreover, $\mathcal{R} = I + S$, while $|S(f)|_{3+\lambda} = O(|f|^2_{3+\lambda})$ where $f \to 0$. This enables us to write Eq. (4.7) in the equivalent form

$$x = -(1 + \beta s)[(L - I)^{-1}(x) + S(L - I)^{-1}(x)] \equiv -(1 + \beta s) A(x) \tag{4.8}$$

The operator A is completely continuous in the sphere $|x|_{3+\lambda} < \delta$ and $A(0) = 0$. Its Fréchet derivative at zero is $(L-I)^{-1}$. According to a theorem by M. A. Krasnosel'skii [4], each simple proper number λ_k of the operator $(L-I)^{-1}$ is a bifurcation point of Eq. (4.8). But between the proper numbers λ_k of the operator $(L-I)^{-1}$ and the simple proper numbers μ_k of the operator L there is an obvious relationship: $1 + \lambda_k = \mu_k$. With the notation $\lambda_k = -(1 + \beta_k s)$ and the use of the expressions for the μ_k, we find the bifurcation values of the parameters β : $\beta_0 = 0$, $\beta_k = s^{-1}(\alpha^2 k^2 - 1)$ $(k = 1, 2, \ldots)$.

For small $x = \eta - 1$ Eq. (4.8) is equivalent to problem (1.1)-(1.9). Taking the fact that the parameter β is positive into account, we write the branching condition for the solution of problem (1.1)-(1.9) in the form

$$\beta_k = s^{-1}(\alpha^2 k^2 - 1), \quad \alpha k > 1 \quad (k = 1, 2, \ldots) \tag{4.9}$$

Condition (4.9) is similar to the branching condition (2.4) and, like the latter, does not contain the Reynolds number. However, a secondary flow that branches from the fundamental for $\beta = \beta_k$ no longer leads to a simple change in the shape of the free surface. It is qualitatively reminiscent of the Taylor vortices that arise in the motion of a liquid between two rotating cylinders.

In the more general case, when $V''(1) \neq 0$, but one of the conditions (4.1) or (4.2) is fulfilled, the considerations leading to the demonstration of the existence of secondary flows remains as before. However, the branching condition (depending on R) has a complicated form and is not given here.

The authors thank V. Kh. Izakson for his discussion of the work.

LITERATURE CITED

1. Chia-Shun Yih, "Instability of a rotating liquid film with a free surface," Proc. Roy. Soc., Ser. A, 258, No. 1292 (1960).
2. L. A. Slobozhanin, "Branching of a cylindrical equilibrium state of a rotating liquid," in: Mathematical Physics, Functional Analysis [in Russian], No. 2, Kharkov (1971).
3. Yu. K. Bratukhin and L. N. Maurin, "Equilibrium shapes of a rotating liquid cylinder," Prikl. Matem. i Mekh., 32, No. 4, (1968).
4. M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations [in Russian], Gostekhizdat, Moscow (1956).